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Self-similar solutions of two-point free boundary problem for heat equation

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1 Introduction and main result

We study the following two-point free boundary problem :

$$\begin{cases} u_t = u_{xx}, \quad -\xi_1(t) < x < \xi_2(t),\ t > 0 \\ u_x(-\xi_1(t), t) = \tan(\theta_1 - \beta_1), \quad u(-\xi_1(t), t) = \xi_1(t)\tan\beta_1, \\ u_x(\xi_2(t), t) = \tan(\beta_2 - \theta_2), \quad u(\xi_2(t), t) = \xi_2(t)\tan\beta_2, \\ u(x, 0) = u_0(x), \quad \xi_1(0) = \xi_{01}, \quad \xi_2(0) = \xi_{02}, \end{cases} \tag{1.1}$$

where β_i and θ_i are given constants satisfying $\beta_i \in [0, \pi/2)$ and $\theta_i \in (0, \beta_i + \pi/2)$, $i = 1, 2$, ξ_{01} and ξ_{02} are positive constants, $u_0 \in C^2[-\xi_{01}, \xi_{02}]$ satisfying the compatibility conditions, and $u_0 > 0$ in $(-\xi_{01}, \xi_{02})$. In this problem (u, ξ_1, ξ_2) are unknown functions to be found.

This type of free boundary problem arises in the combustion theory to describe flame propagation. It is motivated by mathematical modeling of combustion in [1, 5]. Note that the prescribed angle condition at each free boundary makes the problem (1.1) different from the Stefan problem. For a detailed overview of more general or different models we refer the reader to the work of Vazquez [5].

The purpose in this talk is to prove the existence of self-similar solutions for the problem (1.1), which is classified by angle conditions, and also to analyze the stability of them.

The problem (1.1) has fundamental properties as follows. Set

$$\Gamma(t) := \{(x, u(x,t)) \mid -\xi_1(t) < x < \xi_2(t)\} \subset \mathbf{R}^2,$$
$$\partial\Omega_1 := \{(x, z) \mid z = -(\tan\beta_1)x,\ x \leq 0\}, \quad \partial\Omega_2 := \{(x, z) \mid z = (\tan\beta_2)x,\ x \geq 0\}.$$

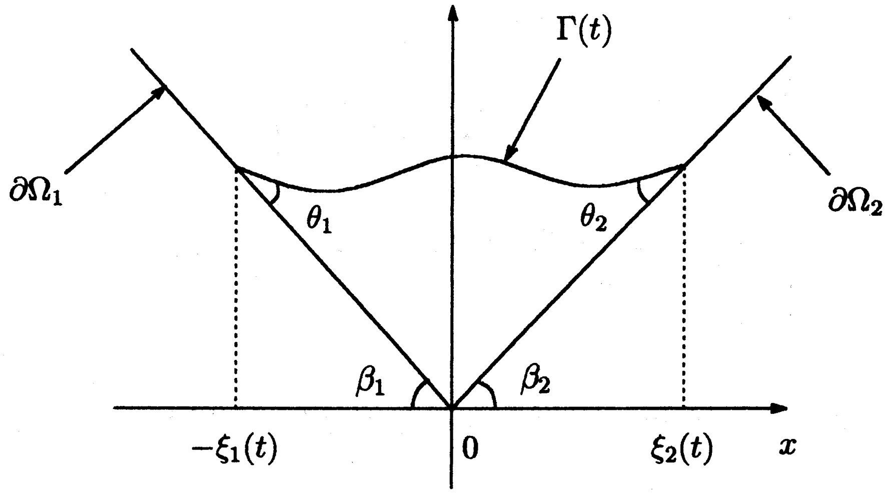


Figure 1: The situation of (1.1)

Let $D(t)$ be the domain enclosed by $\Gamma(t)$, $\partial\Omega_1$, and $\partial\Omega_2$. By a simple calculation, we have

$$\frac{d}{dt}\mu(D(t)) = u_x(\xi_2(t),t) - u_x(-\xi_1(t),t) = \tan(\beta_2-\theta_2) - \tan(\theta_1-\beta_1)$$

where $\mu(D)$ is the area of D. This implies that

$$\frac{d}{dt}\mu(D(t)) \begin{cases} >0 & \text{if} \quad \theta_1+\theta_2<\beta_1+\beta_2, \\ =0 & \text{if} \quad \theta_1+\theta_2=\beta_1+\beta_2, \\ <0 & \text{if} \quad \theta_1+\theta_2>\beta_1+\beta_2. \end{cases} \tag{1.2}$$

It is natural to expect that if $\theta_1+\theta_2<\beta_1+\beta_2$, then $\Gamma(t)$ expands with time t; if $\theta_1+\theta_2=\beta_1+\beta_2$, then $\Gamma(t)$, whose area is preserved in time t, tends to a fixed line as $t\to\infty$; if $\theta_1+\theta_2>\beta_1+\beta_2$, then $\Gamma(t)$ shrinks with time t and vanishes in a finite time $T=T(u_0,\xi_{01},\xi_{02})$.

To analyze the asymptotic behavior of $\Gamma(t)$, we define the following.

Definition 1.1 *(Self-similar) Let $\rho>0$ and set*

$$u^\rho(x,t) := \rho^{-1}u(\rho(x-x_0)+x_0, \rho^2(t-t_0)+t_0).$$

We say that u is self-similar with the center (x_0,t_0) if $u^\rho(x,t)=u(x,t)$ for any $\rho>0$.

Note that the problem (1.1) is invariant for the rescaling $u\mapsto u^\rho$. If u is self-similar and is also a solution of (1.1) for some u_0, ξ_{01}, and ξ_{02}, then we call such u a self-similar solution of (1.1).

There are several references studying self-similar solutions for this type of free boundary problem. For the case $\beta_1=0$ and $\beta_2=\pi/2$, which is one-point free boundary

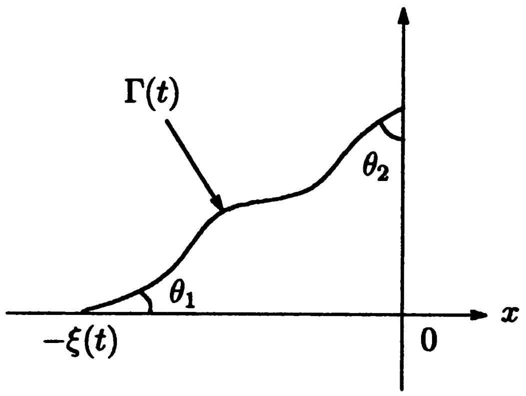


Figure 2: $\beta_1 = 0$ and $\beta_2 = \pi/2$ (see [3, 4])

problem, we refer to [3, 4]. In [4] the author proved the existence and uniqueness of a self-similar solution for a quasilinear parabolic equation $u_t = (a(u_x))_x$ in the case $\theta_1 + \theta_2 < \pi/2$, where $a \in C^2(\mathbf{R})$ and the first derivative of a is positive. The asymptotic stability of this self-similar solution was also obtained. In [3] they considered a focusing problem with $\theta_1 = \pi/4$ and $\theta_2 = \pi/2$. It was proved that a self-similar solution, which vanishes in finite time, exists uniquely and that all solutions are asymptotically equal to this self-similar solution. For the case $\beta_1 = \beta_2 = 0$, we refer to [2]. In [2] they studied in several space dimension and established a theory of existence, uniqueness and regularity for radial symmetric solutions having bounded support. They also investigated the focusing behavior, which is shown to be self-similar, for solutions whose support expands in finite time to fill a hole. We remark that the one-dimensional problem in their model is a special case of our problem.

In order to investigate the existence of self-similar solutions of (1.1), we consider the following problems. Now we set $\alpha_1 := \theta_1 - \beta_1$ and $\alpha_2 := \beta_2 - \theta_2$.

Case $\alpha_1 < \alpha_2$: Analyze ***forward self-similar solutions***. That is, for

$$u(x,t) = \sqrt{2t}v(x/\sqrt{2t}), \quad \xi_1(t) = \sqrt{2t}p, \quad \xi_2(t) = \sqrt{2t}q,$$

we study

$$\begin{cases} v'' + \eta v' - v = 0, & -p < \eta < q, \\ v'(-p) = \tan\alpha_1, & v(-p) = p\tan\beta_1, \\ v'(q) = \tan\alpha_2, & v(q) = q\tan\beta_2. \end{cases} \tag{1.3}$$

In the problem (1.3), v, p, q are unknown function and constants to be found.

Case $\alpha_1 = \alpha_2$: Analyze ***stationaly self-similar solutions***. In this case, a family of the straight lines, namely $u(x) = (\tan\alpha)x + d$ where $\alpha := \alpha_1 = \alpha_2$ and d is any positive constant, is stationary solutions of (1.1).

Case $\alpha_1 > \alpha_2$: Analyze ***backward self-similar solutions***. That is, for

$$u(x,t) = \sqrt{-2t}v(x/\sqrt{-2t}), \quad \xi_1(t) = \sqrt{-2t}p, \quad \xi_2(t) = \sqrt{-2t}q,$$

$$\begin{cases} v'' - \eta v' + v = 0, & -p < \eta < q, \\ v'(-p) = \tan\alpha_1, & v(-p) = p\tan\beta_1, \\ v'(q) = \tan\alpha_2, & v(q) = q\tan\beta_2. \end{cases} \tag{1.4}$$

In the problem (1.4), v, p, q are unknown function and constants to be found.

We are ready to state our main results.

Theorem 1.1 *The following hold:*

(i) Assume that $\alpha_1 < \alpha_2$. Then there exists a unique (up to the translation of time t) forward self-similar solution for (1.1). Moreover, it is asymptotically stable.

(ii) Assume that $\alpha_1 = \alpha_2$. Then there exists a unique stationary self-similar solution for (1.1) with a given D_0, which is the domain enclosed by $\Gamma_0 := \{(x, u_0(x)) | -\xi_{01} \leq x \leq \xi_{02}\}$, $\partial\Omega_1$, and $\partial\Omega_2$.

(iii) Assume that $\alpha_1 > \alpha_2$. Then there is a constant $G_c(< -\tan\beta_1)$ depending only on α_1 and β_1 such that the following hold.

(iii-a) There exists at least one backward self-similar solution for (1.1) if $-\beta_1 \leq \alpha_2 < \alpha_1 \leq \beta_2$.

(iii-b) There exist at least two backward self-similar solutions for (1.1) if $\tan^{-1} G_c < \alpha_2 < -\beta_1 < \alpha_1 \leq \beta_2$.

(iii-c) There exists at least one backward self-similar solution for (1.1) if $\tan^{-1} G_c < \alpha_2 < -\beta_1$ and $\beta_2 < \alpha_1$.

(iii-d) There exists at least one backward self-similar solution for (1.1) if $\bar{\alpha} \leq \alpha_2 \leq \tan^{-1} G_c$ for some $\bar{\alpha} \in (-\pi/2, \alpha_1)$ depending only on α_1, β_1, and β_2.

Remark 1.1 The exact existence for the case (iii) and the stability for the cases (ii), (iii) are still open.

2 Case: $\alpha_1 < \alpha_2$

Give α_1, β_1, p, q, with $\beta_1 \in [0, \pi/2)$, $\alpha_1 \in (-\beta_1, \pi/2)$, $p > 0$, $q > 0$. Let us consider the initial valule problem:

$$\begin{cases} v'' + \eta v' - v = 0, & \eta > -p, \\ v'(-p) = \tan\alpha_1, & v(-p) = p\tan\beta_1. \end{cases} \tag{2.1}$$

Let $F(\eta) = \eta v'(\eta) - v(\eta)$. Then by (2.1) we have $F'(\eta) = -\eta F(\eta)$. It follows that

$$v''(\eta) = -F(\eta) = pA_1 e^{(p^2 - \eta^2)/2}, \quad \eta > -p, \tag{2.2}$$

where $A_1 := \tan\alpha_1 + \tan\beta_1$. Note that $A_1 > 0$, since $\alpha_1 > -\beta_1$. This implies that $v'' > 0$. By an integration of (2.2) from $-p$ to η ($> -p$), we obtain

$$v'(\eta) = \tan\alpha_1 + pA_1 e^{p^2/2}[I^-(p) + I^-(\eta)] \tag{2.3}$$

where

$$I^-(\eta) = \int_0^\eta e^{-s^2/2} ds.$$

Set

$$G(p,q) := v'(q) = \tan\alpha_1 + pA_1 e^{p^2/2}[I^-(p) + I^-(q)].$$

Moreover, by integrating (2.3) from $-p$ to η ($> -p$), we obtain that

$$v(\eta) = \eta\tan\alpha_1 + \eta p A_1 e^{p^2/2}[I^-(p) + I^-(\eta)] + pA_1 e^{(p^2-\eta^2)/2}.$$

Set

$$H(p,q) := \frac{v(q)}{q} = \tan\alpha_1 + pA_1 e^{p^2/2}[I^-(p) + I^-(q)] + \frac{p}{q} A_1 e^{(p^2-q^2)/2}.$$

It is easy to compute that

$$\begin{cases} \dfrac{\partial G}{\partial p}(p,q) = pA_1 + (p^2+1)A_1 e^{p^2/2}[I^-(p) + I^-(q)] \ (>0), \\ \dfrac{\partial G}{\partial q}(p,q) = pA_1 e^{(p^2-q^2)/2} \ (>0), \\ \dfrac{\partial H}{\partial p}(p,q) = pA_1 + (p^2+1)A_1 e^{p^2/2}\left\{[I^-(p) + I^-(q)] + \dfrac{1}{q} e^{-q^2/2}\right\} \ (>0), \\ \dfrac{\partial H}{\partial q}(p,q) = -\dfrac{p}{q^2} A_1 e^{(p^2-q^2)/2} \ (<0). \end{cases}$$

For given $\alpha_2(> \alpha_1)$ and $\beta_2 \in [0, \pi/2)$, we want to solve the equations

$$G(p,q) = \tan\alpha_2 \quad \text{and} \quad H(p,q) = \tan\beta_2. \tag{2.4}$$

for some $p > 0$ and $q > 0$. If we can find the pair of (p,q) satisfying (2.4), (v,p,q) is the solution of (1.3).

Remark 2.1 Clearly $G(p,q) > \tan\alpha_1$. This claims that if $\alpha_1 \geq \alpha_2$, there are no (p,q) satisfying (2.4). That is, there are no forward self-similar solutions of (1.1) for $\alpha_1 \geq \alpha_2$.

Let consider the equation $G(p,q) = \tan\alpha_2$ for a given $\alpha_2(> \alpha_1)$. We first observe that $G(p,q)$ is monotone increasing in p and q. Note that the limit of $I^-(q)$ as $q \uparrow +\infty$ exists and is also finite. Since $G(0,+\infty) = \tan\alpha_1$ and $G(+\infty,+\infty) = +\infty$, there is a unique $p_\infty > 0$ such that

$$G(p_\infty, +\infty) = \tan\alpha_2.$$

In addition, since $G(0,0) = \tan\alpha_1$ and $G(+\infty, 0) = +\infty$, there is a unique $p_0 > 0$ such that

$$G(p_0, 0) = \tan\alpha_2.$$

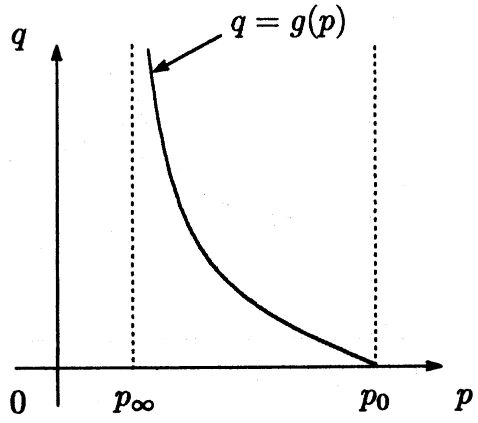


Figure 3: (p, q)-line satisfying $G(p, q) = \tan \alpha_2$

From the monotonisity in p of G, it follows that $p_\infty < p_0$. Then we have for $p \in (p_\infty, p_0)$

$$G(p,0) < G(p_0,0) = \tan \alpha_2 = G(p_\infty, +\infty) < G(p, +\infty)$$

The monotonisity in q of G implies that for each $p \in (p_\infty, p_0)$ there is a unique $q = g(p) > 0$ such that $G(p, g(p)) = \tan \alpha_2$. Note that $g(+\infty) = p_\infty$ and $g(0) = p_0$. Differentiating $G(p, g(p)) = \tan \alpha_2$ with respect to p, we obtain

$$\frac{\partial G}{\partial p}(p, g(p)) + \frac{\partial G}{\partial q}(p, g(p)) \cdot g'(p) = 0.$$

Thus we are led to $g'(p) < 0$, since $\partial G/\partial p > 0$ and $\partial G/\partial q > 0$ (see Figure 3).

Let consider the equation $H(p, q) = \tan \beta_2$ for a given $\beta_2 \in [0, \pi/2)$. We observe that $H(p, q)$ is monotone inscreasing in p for all $q > 0$ and monotone decreasing in q for all $p > 0$. Note that $H(p, +\infty) = G(p, +\infty)$ and $\alpha_2 < \beta_2$. Since $H(p_\infty, 0^+) = +\infty$ and $H(p_\infty, +\infty) = \tan \alpha_2$, there is a unique $\bar{q} > 0$ such that

$$H(p_\infty, \bar{q}) = \tan \beta_2.$$

In addition, since $H(p_\infty, +\infty) = \tan \alpha_2$ and $H(+\infty, +\infty) = +\infty$, there is a unique $p_*(> p_\infty)$ such that

$$H(p_*, +\infty) = \tan \beta_2.$$

Then we have for $p \in (p_\infty, p_*)$

$$H(p, +\infty) < H(p_*, +\infty) = \tan \beta_2 = H(p_\infty, \bar{q}) < H(p, \bar{q}).$$

The monotonisity in q of H implies that for each $p \in (p_\infty, p_*)$ there is a unique $q = h(p) > \bar{q}$ such that $H(p, h(p)) = \tan \beta_2$. Note that $h(p_\infty) = \bar{q}$ and $h(p_*) = +\infty$. Differentiating $H(p, h(p)) = \tan \beta_2$ with respect to p, we derive

$$\frac{\partial H}{\partial p}(p, h(p)) + \frac{\partial H}{\partial q}(p, h(p)) \cdot h'(p) = 0.$$

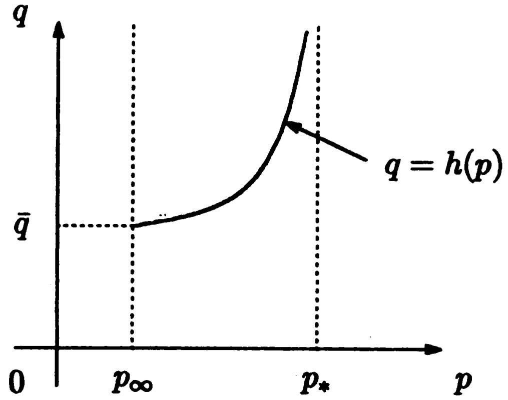


Figure 4: (p,q)-line satisfying $H(p,q) = \tan\beta_2$

Therefore we find $h'(p) > 0$, since $\partial H/\partial p > 0$ and $\partial H/\partial q < 0$ (see Figure 4).
We are ready to state and prove the folloing theorem.

Theorem 2.1 *Assume that $\alpha_1 < \alpha_2$. Then for given $\beta_1, \beta_2 \in [0, \pi/2)$, $\alpha_1 \in (-\beta_1, \pi/2)$, and $\alpha_2 \in (\alpha_1, \beta_2)$ there is a unique solution (v, p, q) to the problem (1.3).*

Proof. Combine Figure 3 and Figure 4. Then we see that there is a unique (p, q) with $p \in (p_\infty, \bar{p})$, where $\bar{p} := \min\{p_0, p_*\}$, and $q > \bar{q}$ such that $g(p) = h(p) = q$. This prove the theorem. □

We state the following theorem withou the proof.

Theorem 2.2 *(Stability of a forward self-similar solution) Assume that $\alpha_1 < \alpha_2$. Also assume that $u_0 \in C^2[-\xi_{01}, \xi_{02}]$ satisfies the compatibility conditions and $u_0 > 0$ in $(-\xi_{01}, \xi_{02})$. Let $\Gamma(t)$ be a smooth solution of (1.1) with the initial data $\Gamma_0 = \{(x, u_0(x)) \mid -\xi_{01} \le x \le \xi_{02}\}$ and $S(t)$ be a forward self-similar solution denoted as*

$$S(t) := \{(\hat{x}, \sqrt{2t}v(\hat{x}/\sqrt{2t})) \mid -\sqrt{2t}p \le \hat{x} \le \sqrt{2t}q\}$$

where (v, p, q) is a solution of (1.3). Then $S(t)$ is asymptotically stable in the sense :

$$d_H(\Gamma(t), S(t)) \le Ct^{-\delta}, \quad t > 1,$$

for some $\delta \in (0, 1/2)$ and a constant $C(> 0)$, which depends on the initial data Γ_0. Here d_H denotes the Hausdorff distance.

To prove this theorem, we construct a sub-solution and a super-solution, which converge to $S(t)$ asymptotically as $t \to \infty$, and apply the strong maximum principle.

3 Case: $\alpha_1 = \alpha_2$

In this case, there is a family of *stationary self-similar solutions* of (1.1), that is,

$$u_d(x,t) = u_d(x) = (\tan\alpha)x + d,$$

where $\alpha := \alpha_1 = \alpha_2$ and d is any positive constant. The corresponding fixed end points to u_d are given by

$$p = \frac{d}{\tan\alpha_1 + \tan\beta_1}, \quad q = \frac{d}{\tan\beta_2 - \tan\alpha_2}.$$

According to (1.2), the condition $\alpha_1 = \alpha_2$ implies the area-preserving property. Let D_0 be the domain enclosed by $\Gamma_0 := \{(x, u_0(x)) | -\xi_{01} \le x \le \xi_{02}\}$, $\partial\Omega_1$, and $\partial\Omega_2$. Set $A_1 := \tan\alpha_1 + \tan\beta_1$ and $A_2 := \tan\beta_2 - \tan\alpha_2$. Let

$$d_* = \sqrt{\frac{2A_1A_2}{A_1 + A_2}\mu(D_0)}.$$

Then a stationary self-similar solution of (1.1) is uniquely determined as $u_{d_*}(x) = ax + d_*$ for a given D_0.

4 Case: $\alpha_1 > \alpha_2$

Give α_1, β_1, p, q, with $\beta_1 \in [0, \pi/2)$, $\alpha_1 \in (-\beta_1, \pi/2)$, $p > 0$, $q > 0$. Let us consider the initial valule problem:

$$\begin{cases} v'' - \eta v' + v = 0, & \eta > -p, \\ v'(-p) = \tan\alpha_1, & v(-p) = p\tan\beta_1. \end{cases} \tag{4.1}$$

Then as before we have

$$v''(\eta) = -pA_1 e^{-(p^2-\eta^2)/2}, \quad \eta > -p, \tag{4.2}$$

where $A_1 = \tan\alpha_1 + \tan\beta_1 > 0$. Note that $v'' < 0$. By an integration of (4.2) from $-p$ to η $(> -p)$, we obtain

$$v'(\eta) = \tan\alpha_1 - pA_1 e^{-p^2/2}[I^+(p) + I^+(\eta)], \quad \eta > -p, \tag{4.3}$$

where

$$I^+(\eta) = \int_0^\eta e^{s^2/2} ds.$$

Set

$$\hat{G}(p, q) := v'(q) = \tan\alpha_1 - pA_1 e^{-p^2/2}[I^+(p) + I^+(q)].$$

In addition, by integrating (4.3) again, we derive that

$$v(\eta) = \eta\tan\alpha_1 - \eta pA_1 e^{-p^2/2}[I^+(p) + I^+(\eta)] + pA_1 e^{-(p^2-\eta^2)/2}.$$

Also set

$$\hat{H}(p, q) := \frac{v(q)}{q} = \tan\alpha_1 - pA_1 e^{-p^2/2}[I^+(p) + I^+(q)] + \frac{p}{q}A_1 e^{-(p^2-q^2)/2}.$$

It is easy to compute that

$$\begin{cases} \dfrac{\partial \hat{G}}{\partial p}(p,q) = -pA_1 + (p^2-1)A_1 e^{-p^2/2}[I^+(p) + I^+(q)], \\ \dfrac{\partial \hat{G}}{\partial q}(p,q) = -pA_1 e^{-(p^2-q^2)/2} \ (<0), \\ \dfrac{\partial \hat{H}}{\partial p}(p,q) = -pA_1 + (p^2-1)A_1 e^{-p^2/2}\left\{[I^+(p) + I^+(q)] - \dfrac{1}{q}e^{q^2/2}\right\}, \\ \dfrac{\partial \hat{H}}{\partial q}(p,q) = -\dfrac{p}{q^2}A_1 e^{-(p^2-q^2)/2} \ (<0). \end{cases}$$

For given $\alpha_2 (< \alpha_1)$ and $\beta_2 \in [0, \pi/2)$, we want to solve the equations

$$\hat{G}(p,q) = \tan\alpha_2 \quad \text{and} \quad \hat{H}(p,q) = \tan\beta_2. \tag{4.4}$$

for some $p > 0$ and $q > 0$. If we can find the pair of (p,q) satisfying (4.4), (v,p,q) is the solution of (1.4).

Remark 4.1 Clearly $\hat{G}(p,q) < \tan\alpha_1$. This claims that if $\alpha_1 \le \alpha_2$, there are no (p,q) satisfying (4.4). That is, there are no backward self-similar solutions of (1.1) for $\alpha_1 \le \alpha_2$.

In order to solve (4.4), let study the fuctions $\hat{G}(p,q)$ and $\hat{H}(p,q)$. Now set

$$J(p) := \frac{p}{p^2-1}e^{p^2/2} - I^+(p) \quad \text{for} \quad p \ne 1,$$
$$K(q) := I^+(q) - \frac{1}{q}e^{q^2/2} \quad \text{for} \quad q > 0.$$

We compute that

$$J'(p) = -\frac{1}{(p^2-1)^2}e^{p^2/2} < 0 \quad \text{for} \quad p \ne 1, \tag{4.5}$$
$$K'(q) = \frac{1}{q^2}e^{q^2/2} > 0 \quad \text{for} \quad q > 0, \tag{4.6}$$

and observe that

$$J(0) = 0, \quad J(1^-) = -\infty, \quad J(1^+) = +\infty, \quad J(+\infty) = -\infty, \tag{4.7}$$
$$K(0^+) = -\infty, \quad K(+\infty) = +\infty. \tag{4.8}$$

It follows from (4.6) and (4.8) that there is a unique $r_0 > 0$ such that

$$K(q) \begin{cases} < 0 & \text{if} \quad 0 < q < r_0, \\ = 0 & \text{if} \quad q = r_0, \\ > 0 & \text{if} \quad q > r_0. \end{cases}$$

First we study the function $\hat{G}(p,q)$. Note that

$$\begin{cases} \dfrac{\partial \hat{G}}{\partial p}(p,q) < 0 \quad \text{for} \ \ p \in (0,1], \\ \dfrac{\partial \hat{G}}{\partial p}(p,q) = A_1(p^2-1)e^{-p^2/2}[I^+(q) - J(p)] \quad \text{for} \ \ p > 1. \end{cases}$$

Since $(I^+)'(q) > 0$, $I^+(0) = 0$, and $I^+(+\infty) = +\infty$, there is a unique $p_c(q) > 1$ such that $J(p_c(q)) = I^+(q)$ for each $q > 0$. We have $p_c(0^+) \in (1, +\infty)$, $p_c(+\infty) = 1$, and $p_c'(q) < 0$. These imply that for all $q > 0$

$$\frac{\partial \hat{G}}{\partial p}(p,q) \begin{cases} < 0 & \text{if} \ \ p < p_c(q); \\ = 0 & \text{if} \ \ p = p_c(q); \\ > 0 & \text{if} \ \ p > p_c(q). \end{cases} \tag{4.9}$$

Then we find

$$G_c := \hat{G}(p_c(0), 0) = -\tan\beta_1 - \frac{1}{p_c^2(0) - 1}(\tan\alpha_1 + \tan\beta_1) \ (< -\tan\beta_1). \tag{4.10}$$

Next we study the function $\hat{H}(p,q)$. Note that

$$\begin{cases} \dfrac{\partial \hat{H}}{\partial p}(p,q) A_1(p^2-1)e^{-p^2/2}[K(q) - J(p)] \quad \text{for} \ \ p \neq 1, \\ \dfrac{\partial \hat{H}}{\partial p}(1,q) = -A_1 < 0. \end{cases}$$

Consider the case $0 < p < 1$. For $q \geq r_0$

$$\frac{\partial \hat{H}}{\partial p}(p,q) = -pA_1 + A_1(p^2-1)e^{-p^2/2}[K(q) + I^+(p)] < 0. \tag{4.11}$$

On the other hand, by virtue of (4.5) and (4.7), we see that $J(p) < 0$ for $p \in (0,1)$. This implies that for each $p \in (0,1)$ there exists a unique $q_s(p) \in (0, r_0)$ such that

$$K(q_s(p)) = J(p).$$

Note that $q_s(0^+) = r_0$, $q_s(1^-) = 0$, and $q_s'(p) < 0$. Thus we derive for $0 < q < r_0$

$$\frac{\partial \hat{H}}{\partial p}(p,q) \begin{cases} > 0 & \text{if} \ \ 0 < q < q_s(p); \\ = 0 & \text{if} \ \ q = q_s(p); \\ < 0 & \text{if} \ \ q > q_s(p) \end{cases} \tag{4.12}$$

Consider the case $p > 1$. It follows from (4.5)-(4.8) that there exists a unique $q_u(p) > 0$ such that

$$K(q_u(p)) = J(p).$$

Note that $q_u(1^+) = +\infty$, $q_u(+\infty) = 0$, and $q_u'(p) < 0$. Therefore we are led to

$$\frac{\partial \hat{H}}{\partial p}(p,q) \begin{cases} < 0 & \text{if} \quad 0 < q < q_u(p); \\ = 0 & \text{if} \quad q = q_u(p); \\ > 0 & \text{if} \quad q > q_u(p). \end{cases} \tag{4.13}$$

Let consider the equation $\hat{G}(p,q) = \tan\alpha_2$ for a given $\alpha_2(< \alpha_1)$. We separate into three cases; $(a) - \beta_1 \leq \alpha_2$, $(b) \tan^{-1} G_c < \alpha_2 < -\beta_1$, $(c) \alpha_2 \leq \tan^{-1} G_c$. For the sake of convenience, we analyze them in order of $(c) \to (b) \to (a)$.

Case $\alpha_2 \leq \tan^{-1} G_c$: If $\alpha_2 = \tan^{-1} G_c (= \tan^{-1}[\hat{G}(p_c(0),0)])$, $(p_c(0),0)$ is a solution of $\hat{G}(p,q) = \tan\alpha_2$. Thus we study the case $\alpha_2 < \tan^{-1} G_c$. Recalling (4.9), we have $\tan\alpha_2 < G_c = \hat{G}(p_c(0),0) \leq \hat{G}(p,0)$ for all $p > 0$. We also find $\hat{G}(p,+\infty) = -\infty$. It follows from $\partial\hat{G}/\partial q < 0$ that for each $p > 0$ there is a unique $q = \hat{g}(p) > 0$ such that $\hat{G}(p,\hat{g}(p)) = \tan\alpha_2$. Note that for $(\tilde{p},\tilde{q})$ on the line $p = p_c(q)$ we also have $\tilde{q} = \hat{g}(\tilde{p}) \in (0,+\infty)$ satifying $\hat{G}(\tilde{p},\hat{g}(\tilde{p})) = \tan\alpha_2$. Differentiating $\hat{G}(p,\hat{g}(p)) = \tan\alpha_2$ with respect to p, we obtain

$$\frac{\partial \hat{G}}{\partial p}(p,\hat{g}(p)) + \frac{\partial \hat{G}}{\partial q}(p,\hat{g}(p)) \cdot \hat{g}'(p) = 0.$$

Recalling (4.9) again, this implies that

$$\begin{cases} \hat{g}'(p) < 0 & \text{for} \quad 0 < p < \tilde{p}, \\ \hat{g}'(p) > 0 & \text{for} \quad p > \tilde{p}. \end{cases}$$

In addition, using the reduction to absurdity, we see $g(0^+) = +\infty$ and $g(+\infty) = +\infty$ (see Figure 5(c))

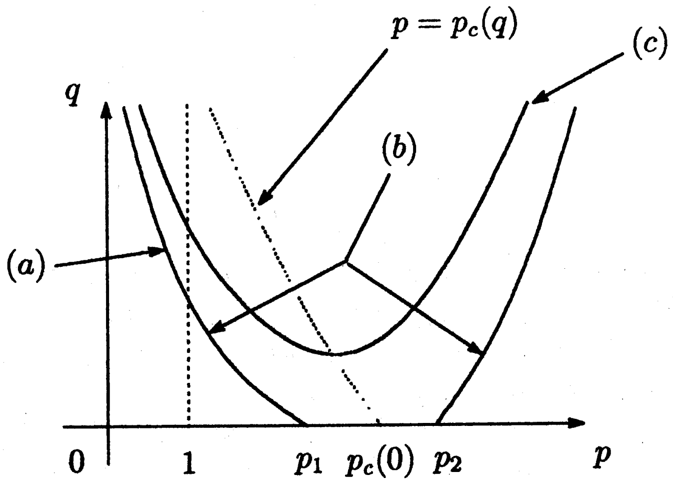
Case $\tan^{-1} G_c < \alpha_2 < -\beta_1$: We observe that $\hat{G}(0,0) = \tan\alpha_1 (> \tan\alpha_2)$ and

$$\begin{aligned} \hat{G}(p,0) &= \tan\alpha_1 - pA_1 e^{-p^2/2} I^+(p) \\ &\to \tan\alpha_1 - A_1 = -\tan\beta_1 (> \tan\alpha_2) \quad \text{as} \quad p \to +\infty. \end{aligned}$$

Then it follows from (4.9) that there exist $p_1 \in (0, p_c(0))$ and $p_2 \in (p_c(0), +\infty)$ such that $\hat{G}(p_1,0) = \hat{G}(p_2,0) = \tan\alpha_2$. For $p \in (p_1,p_2)$, we obatin that $\hat{G}(p,0) < \tan\alpha_2$. In view of $\partial\hat{G}/\partial q < 0$, we are led to $\hat{G}(p,q) < \tan\alpha_2$ for $p \in (p_1,p_2)$ and $q > 0$. Thus for $p \in (p_1,p_2)$ there is no solution of $\hat{G}(p,q) = \tan\alpha_2$. For $p \in (0,p_1) \cup (p_2,+\infty)$, we observe that $\hat{G}(p,0) > \tan\alpha_2$. Since $\hat{G}(p,+\infty) = -\infty$ and $\partial\hat{G}/\partial q < 0$ for all $p > 0$, there is a unique $q = \hat{g}(p) > 0$ such that $\hat{G}(p,\hat{g}(p)) = \tan\alpha_2$ for each $p \in (0,p_1) \cup (p_2,+\infty)$. Appying the same argument as the case $\alpha_2 \leq \tan^{-1} G_c$, we see

$$\begin{cases} \hat{g}'(p) < 0 & \text{for} \quad 0 < p < p_1, \\ \hat{g}'(p) > 0 & \text{for} \quad p > p_2. \end{cases}$$

We also have $g(0^+) = +\infty$ and $g(+\infty) = +\infty$. Moreover, by means of $\hat{G}(p_1,0) = \hat{G}(p_2,0) = \tan\alpha_2$, we derive $\hat{g}(p_1) = \hat{g}(p_2) = 0$ (see Figure 5(b)).



$(a) - \beta_1 \leq \alpha_2$ $(b) \tan^{-1} G_c < \alpha_2 < -\beta_1$ $(c) \alpha_2 \leq \tan^{-1} G_c$

Figure 5: (p, q)-line satisfying $\hat{G}(p, q) = \tan \alpha_2$

<u>Case $-\beta_1 \leq \alpha_2$</u>: Let $p_1 \in (0, p_c(0))$ be defined as the above. Recalling $\hat{G}(p, 0) \to -\tan \beta_1$ as $p \to +\infty$ with $\partial \hat{G}/\partial p > 0$ for $p > p_c(0)$, we have $\hat{G}(p, 0) < \tan \alpha_2$ for $p > p_1$. It follows from $\partial \hat{G}/\partial q < 0$ that $\hat{G}(p, q) < \tan \alpha_2$ for $p > p_1$ and $q > 0$. Thus for $p > p_1$ there is no solution of $\hat{G}(p, q) = \tan \alpha_2$. For $p \in (0, p_1)$, we derive that $\hat{G}(p, 0) > \tan \alpha_2$. Since $\hat{G}(p, +\infty) = -\infty$ and $\partial \hat{G}/\partial q < 0$ for all $p > 0$, there is a unique $q = \hat{g}(p) > 0$ such that $\hat{G}(p, \hat{g}(p)) = \tan \alpha_2$ for each $p \in (0, p_1)$. Appying the same argument as the previous case, we derive $\hat{g}'(p) < 0$ for $p \in (0, p_1)$, $\hat{g}(0^+) = +\infty$, and $g(p_1) = 0$ (see Figure 5(a)).

Let consider the equation $\hat{H}(p, q) = \tan \beta_2$ for a given $\beta_2 \in [0, \pi/2)$. Since $\hat{H}(p, 0^+) = +\infty$ and $\hat{H}(p, +\infty) = -\infty$ for all $p > 0$, it follows from $\partial \hat{H}/\partial q < 0$ that for each $p > 0$ there is a unique $q = \hat{h}(p) > 0$ such that $\hat{H}(p, \hat{h}(p)) = \tan \beta_2$. Now we compute that

$$\hat{H}(p, q_u(p)) = -\tan \beta_1 - \frac{1}{p^2 - 1} A_1 \ (< 0) \quad \text{for} \quad p > 1, \tag{4.14}$$

$$\hat{H}(p, q_s(p)) = -\tan \beta_1 + \frac{1}{1 - p^2} A_1, \quad \text{for} \quad p \in (0, 1). \tag{4.15}$$

It follows from (4.14), $\tan \beta_2 > 0$, and $\partial \hat{H}/\partial q < 0$ that $\hat{h}(p) \in (0, q_u(p))$ for $p > 1$. Then, in view of $\partial \hat{H}/\partial q < 0$ and (4.13), we have $\hat{h}'(p) < 0$ for $p > 1$. Note that $\hat{h}(1) \in (0, +\infty)$ and $\hat{h}(+\infty) = 0$, since $q_u(1^+) = +\infty$ and $q_u(+\infty) = 0$. Hereafter, we investigate $\hat{h}(p)$ for $p \in (0, 1)$. By (4.15), $\hat{H}(p, q_s(p)) = \tan \beta_2$ is equivalent to

$$p^2 = \frac{\tan \beta_2 - \tan \alpha_1}{\tan \beta_1 + \tan \beta_2} \in (0, 1). \tag{4.16}$$

We separate into three cases; $(\bar{a}) \beta_2 < \alpha_1$, $(\bar{b}) \beta_2 = \alpha_1$, $(\bar{c}) \beta_2 > \alpha_1$.

Case $\beta_2 < \alpha_1$: Note that there is no $p \in (0, +\infty)$ satisfying (4.16). Since $\tan\beta_1 + \tan\beta_2 \leq A_1$, we have

$$\begin{aligned}\hat{H}(p, \hat{h}(p)) = \tan\beta_2 &\leq -\tan\beta_1 + A_1 \\ &< -\tan\beta_1 + \frac{1}{1-p^2}A_1 = \hat{H}(p, q_s(p)) \quad \text{for } p \in (0,1).\end{aligned}$$

Recalling that $\hat{H}(p,q)$ is monotone decreasing in q, we see $\hat{h}(p) > q_s(p)$ for all $p \in (0,1)$. It follows from $\partial\hat{H}/\partial q < 0$, (4.11), and (4.12) that $\hat{h}'(p) < 0$ for all $p \in (0,1)$. In addition, we derive $\hat{h}(0^+) = +\infty$ (see Figure 6). Indeed, if $\hat{h}(0^+) < +\infty$, for any $q_\star > \max\{2\hat{h}(0^+), r_0\}$ we have $\hat{H}(p, q_\star) \to \tan\alpha_1$ as $p \to 0^+$. Then (4.11) implies that there is a $p_\star \in (0,1)$ such that $\hat{H}(p, q_\star) > \tan\beta_2 = \hat{H}(p, \hat{h}(p))$ for all $p < p_\star$. It follows from $\partial\hat{H}/\partial q < 0$ that $q_\star < \hat{h}(p)$ for all $p < p_\star$. This is a contradiction. Hence $\hat{h}(0^+) = +\infty$.

Case $\beta_2 = \alpha_1$: Applying the same argument as the previous case, we have $\hat{h}(p) > q_s(p)$ for all $p \in (0,1)$ and $\hat{h}'(p) < 0$. Moreover, since $\hat{H}(p, r_0) < \tan\alpha_1 = \tan\beta_2 = \hat{H}(p, \hat{h}(p))$ and $\partial\hat{H}/\partial q < 0$ imply that $\hat{h}(p) \in (q_s(p), r_0)$ for all $p \in (0,1)$, we see $\hat{h}(0^+) = r_0$ (see Figure 6).

Case $\beta_2 > \alpha_1$: There is a unique $p_\dagger \in (0,1)$ satisfying (4.16). That is, $\hat{H}(p_\dagger, q_s(p_\dagger)) = \tan\beta_2$. Using (4.12) and $\partial\hat{H}/\partial q < 0$, it is easy to see that

$$\begin{cases} \hat{h}(p) < q_s(p) \quad \text{and} \quad \hat{h}'(p) > 0 \quad \text{for} \quad 0 < p < p_\dagger, \\ \hat{h}(p) > q_s(p) \quad \text{and} \quad \hat{h}'(p) < 0 \quad \text{for} \quad p_\dagger < p < 1. \end{cases}$$

Note that $\hat{h}(0^+) \in [0, r_0)$ (see Figure 6).

From now on, we assume $\beta_1, \beta_2 \in [0, \pi/2)$, $\alpha_1 \in (-\beta_1, \pi/2)$, and $\alpha_2 \in (-\pi/2, \alpha_1) \cap (-\pi/2, \beta_2)$. We are ready to state and prove the folloing theorems.

Theorem 4.1 *Assume that $-\beta_1 \leq \alpha_2 < \alpha_1 \leq \beta_2$. Then there is at least one solution to the problem (1.4). Assume that $\tan^{-1} G_c < \alpha_2 < -\beta_1 < \alpha_1 \leq \beta_2$, Then there are at least two solutions to the problem (1.4).*

Theorem 4.2 *Assume that $\tan^{-1} G_c < \alpha_2 < -\beta_1$ and $\beta_2 < \alpha_1$. Then there is at least one solution to the problem (1.4).*

Proof of Theorem 4.1 and 4.2. For the first half of Theorem 4.1, combine Figure 5(a) and Figure 6($\bar{b}$), ($\bar{c}$). For the second half of Theorem 4.1, combine Figure 5(b) and Figure 6 ($\bar{b}$), ($\bar{c}$). For Theorem 4.2, combine Figure 5(b) and Figure 6($\bar{a}$). □

Theorem 4.3 *Assume that $\alpha_2 \leq \tan^{-1} G_c$. Then there exists $\alpha_* \in (-\pi/2, \alpha_1)$ depending only on α_1, β_1, and β_2 such that the problem (1.4) has at least on solution if $\alpha_2 \geq \alpha_*$.*

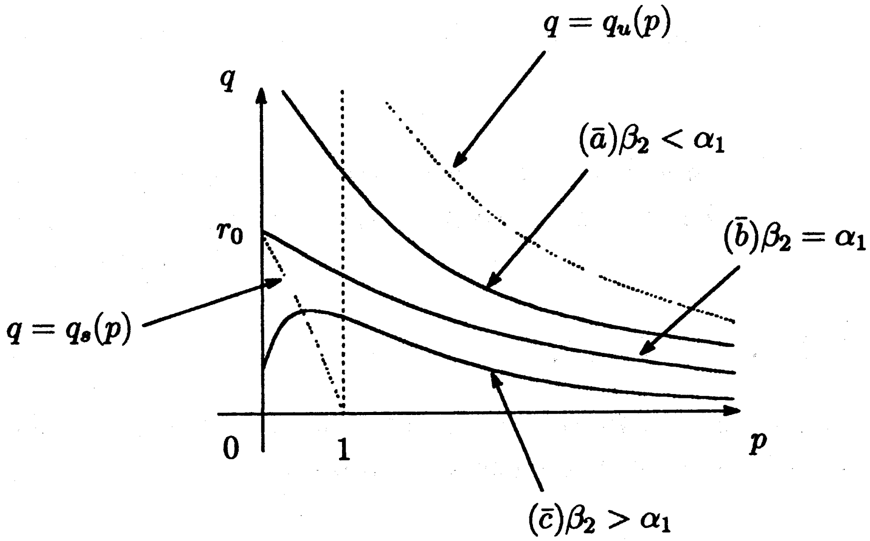


Figure 6: (p, q)-line satisfying $\hat{H}(p, q) = \tan\beta_2$

Proof. Recall that for $(\tilde{p}, \tilde{q})$ on the line $p = p_c(q)$ we have $\tilde{q} = \hat{g}(\tilde{p}) \in (0, +\infty)$ satisfying $\hat{G}(\tilde{p}, \hat{g}(\tilde{p})) = \tan\alpha_2$. This is also written as $\hat{G}(p_c(\tilde{q}), \tilde{q}) = \tan\alpha_2$. Since

$$\hat{G}(p_c(\tilde{q}), \tilde{q}) = -\tan\beta_1 - \frac{1}{p_c^2(\tilde{q}) - 1} A_1$$

is monotone decreasing in $\tilde{q}$, the function $\tilde{q} = \tilde{q}(\alpha_2)$ is monotone decreasing as α_2 increases. Note that $\tilde{q}(\alpha_2) \to +\infty$ as $\alpha_2 \to -\pi/2$ and $\tilde{q}(\alpha_2) \to 0$ as $\alpha_2 \to \tan^{-1} G_c$. Combining this fact and Figure 6, we see that there exists $\alpha_* \in (-\pi/2, \alpha_1)$ such that the problem (1.4) has at least on solution if $\alpha_2 \geq \alpha_*$. □

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